

GENERALIZED UNCERTAINTY PRINCIPLES

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ABSTRACT. The phenomenon in the essence of classical uncertainty principles is well known since the thirties of the last century. We introduce a new phenomenon which is in the essence of a new notion that we introduce: "Generalized Uncertainty Principles". We show the relation between classical uncertainty principles and generalized uncertainty principles. We generalized "Landau-Pollak-Slepian" uncertainty principle. Our generalization relates the following two quantities and two scaling parameters: 1) The weighted time spreading $\int_{-\infty}^{\infty} |f(x)|^2 w_1(x) dx$, ($w_1(x)$ is a non-negative function). 2) The weighted frequency spreading $\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 w_2(\omega) d\omega$. 3) The time weight scale a , $w_{1a}(x) = w_1(xa^{-1})$ and 4) The frequency weight scale b , $w_{2b}(\omega) = w_2(\omega b^{-1})$. "Generalized Uncertainty Principle" is an inequality that summarizes the constraints on the relations between the two spreading quantities and two scaling parameters. For any two reasonable weights $w_1(x)$ and $w_2(\omega)$, we introduced a three dimensional set in R^3 that is in the essence of many uncertainty principles. The set is called "possibility body". We showed that classical uncertainty principles (such as the Heisenberg-Pauli-Weyl uncertainty principle) stem from lower bounds for different functions defined on the possibility body. We investigated qualitative properties of general uncertainty principles and possibility bodies. Using this approach we derived new (quantitative) uncertainty principles for Landau-Pollak-Slepian weights. We found the general uncertainty principles related to homogeneous weights, $w_1(x) = w_2(x) = x^k$, $k \in N$, up to a constant.

1. NOTATIONS

- We denote by L^2 the space of functions such that: $\int_{-\infty}^{\infty} |f(x)|^2 dx < \infty$ with the inner product defined by: $\langle f, g \rangle = \int_{-\infty}^{\infty} f(x) \overline{g(x)} dx$ and norm defined by: $\|f\| = \langle f, f \rangle^{\frac{1}{2}}$
- Fourier transform: $\hat{f}(\omega) = \int_{R^d} f(x) e^{-2\pi i x \cdot \omega} dx$
- Translation operators: $T_a f(x) = f(x - a)$

Date: 9 July 2008.

Key words and phrases. Uncertainty principles, Time-Frequency analysis, Prolate spheroidal wave functions.

- Modulation operators: $M_b f(x) = \exp(ibx)f(x)$
- Scaling operators: $S_a f(x) = f(xa^{-1})$
- Time limiting operators $D_h : L^2 \rightarrow L^2$: $\widehat{D_h f}(\omega) = \int_{-h}^h f(x)e^{-2\pi i x \cdot \omega} dx$
- Band limiting operators $B_m : L^2 \rightarrow L^2$: $B_m f(x) = \int_{-m}^m \hat{f}(\omega)e^{-2\pi i x \cdot \omega} d\omega$
- The function $\cos^{-1}(x) : [-1, 1] \rightarrow [0, \pi]$ is the inverse of $\cos(x)$:
 $\cos^{-1}(\cos(x)) = x, \forall x \in [0, \pi]$
- The function $\sin^{-1}(x) : [-1, 1] \rightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$ is the inverse of $\sin(x)$:
 $\sin^{-1}(\sin(x)) = x, \forall x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

2. INTRODUCTION

The meta-principle that a signal can not be localized both at time and frequency is reflected as inequalities involving a function, say f , and its Fourier transform, \hat{f} . We will call that kind of inequalities *classical uncertainty principles*. A good survey for the subject by Gerald B. Folland and Alladi Sitaram is [4]. Uncertainty principles uses the notion of concentration (e.g. Landau-Pollak-Slepian (LPS), see below) or the notion of spreading (e.g. Heisenberg-Pauli-Weyl (HPW), see below). By "translating" Landau and Pollak (LP) result from "concentration language" to "spreading language" on one side and adding two parameters to HPW result on the other side, we show that those results are special cases (up to minor changes in the LP case) of what we call *Generalized Uncertainty Principles*. Our approach explains the qualitative behavior which is related to LP result and HPW result. The important quantitative results in LP result and HPW result can not be achieved using our approach. recall the (classical) Heisenberg-Pauli-Weyl uncertainty principle:

Theorem 2.1 (Heisenberg-Pauli-Weyl Uncertainty Principle). *If $f \in L^2(R)$ and $a, b \in R$ are arbitrary, then*

$$(1) \quad \left(\int_{-\infty}^{\infty} (x-a)^2 |f(x)|^2 dx \right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} (\omega-b)^2 |\hat{f}(\omega)|^2 d\omega \right)^{\frac{1}{2}} \geq \frac{1}{4\pi} \|f\|_2^2$$

Equality holds if and only if f is a multiple of $\exp^{2\pi i b(x-a)} \exp^{-\pi(x-a)^2/c}$ for some $a, b \in R$ and $c > 0$

□

HPW uncertainty principle was the first uncertainty principle that appeared ([7, 10] [16, Appendix 1]). Different variations of HPW uncertainty principle have been published since then [4, section 3]. LP [11] defined for every $f \in L^2$ its *time concentration* by:

$$(2) \quad \alpha_T(f) = \frac{\left(\int_{-T}^T |f(x)|^2 dx \right)^{\frac{1}{2}}}{\left(\int_{-\infty}^{\infty} |f(x)|^2 dx \right)^{\frac{1}{2}}}$$

and its *frequency concentration* by:

$$(3) \quad \beta_\Omega(f) = \frac{\left(\int_{-\Omega}^{\Omega} |\hat{f}(\omega)|^2 d\omega \right)^{\frac{1}{2}}}{\left(\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 d\omega \right)^{\frac{1}{2}}}$$

and found explicitly the following set of points in R^2 :

$$M_{\Omega,T} = \bigcup_{f \in L^2} (\alpha_T(f), \beta_\Omega(f))$$

We define the complement of $M_{\Omega,T}$ in

$$D_{LP} = [0, 1] \times [0, 1] \setminus \{(0, 1), (1, 0), (1, 1)\}:$$

$$M'_{\Omega,T} = D_{LP} \setminus M_{\Omega,T}$$

We will call the pair $(M_{\Omega,T}, M'_{\Omega,T})$ a "possibility map"; where $M_{\Omega,T}$ is the "possible area" and $M'_{\Omega,T}$ is the "impossible area" (note the dependence on Ω and T). Of course, for individual $f \in L^2$, the point $(\alpha_T(f), \beta_\Omega(f))$ depends on T and Ω . LP noticed that the "possibility map" depends on the product $c = \Omega T$ only and not on Ω and T separately ($M_c = M_{\Omega,T}$, In [11] the integral bounds in the numerator of (2) are from $-\frac{T}{2}$ to $\frac{T}{2}$ and therefore the notation in their papers is $c = \Omega \frac{T}{2}$). LP [11] proved an uncertainty principle of the form:

$$(4) \quad c \geq \phi(\alpha, \beta)$$

$$\phi : D_{LP} \rightarrow [0, \infty)$$

(more details is section 3). We will call inequalities of the type (4) and its generalizations that we will introduce below "Generalized uncertainty principles". We will call the pair (M_c, M'_c) "the possibility map of level c ", where M_c is the "possible area of level c " and M'_c is the "impossible area of level c ". We will call the set

$$PB = \{(\alpha, \beta, c) | (\alpha, \beta) \in M_c, c \in (0, \infty)\}$$

"a possibility body". In section 3 we state LP result in the original way and in a way that can be generalized. We show how uncertainty principles (inequalities) are derived by bounding functions which are defined on the "possible area of level c " (where c is a parameter in the inequality). We also show that in one natural coordinate system the "possible area of level c " is convex and in another natural coordinate system the "possible area of level c " is non-convex.

In section 4 we show that for some general weights we have the same qualitative behavior regarding uncertainty, and using HPW uncertainty principle we derive the possibility map and possibility body for the HPW weight (x^2) . In section 5 we discuss the question of convexity of the possibility body and the question of the right coordinate system for describing the "General Uncertainty Principle" phenomenon. We find the general uncertainty principles for homogenous weights $w_1(x) = w_2(x) = x^k$, $k \in \mathbb{N}$. *Heisenberg-Pauli-Weyl general uncertainty principle is a special case that corresponds to $k = 2$.*

3. SLEPIAN-POLLAK-LANDAU UNCERTAINTY PRINCIPLE

In a series of papers by LPS [11, 12, 13] the following integral equation was investigated:

$$(5) \quad \lambda f(t) = \frac{1}{\pi} \int_{-T}^T f(s) \frac{\sin \Omega(t-s)}{t-s} ds$$

on $L^2[-T, T]$

They have showed that the eigenvalues of (5) are distinct, positive and depend on the product $c = \Omega T$. (In [11] the integral bounds of (5) are from $-\frac{T}{2}$ to $\frac{T}{2}$ and therefore the notation in their papers is $2c = \Omega T$).

LP [11] proved the following uncertainty principle:

Theorem 3.1 (LP Theorem). *There is a function f such that $\|f\| = 1$, $\alpha_T(f) = \alpha$ and $\beta_\Omega(f) = \beta$, under the following conditions, and only under the following conditions:*

- 1) If $\alpha = 0$ and $0 \leq \beta < 1$
- 2) If $0 < \alpha < \sqrt{\lambda_0}$ and $0 \leq \beta \leq 1$
- 3) If $\sqrt{\lambda_0} \leq \alpha < 1$ and $\cos^{-1} \alpha + \cos^{-1} \beta \geq \cos^{-1} \sqrt{\lambda_0}$
- 4) If $\alpha = 1$ and $0 < \beta \leq \sqrt{\lambda_0}$

where λ_0 is the largest eigenvalue of (5)

□

The complexity of computing the biggest eigenvalue, λ_0 , of (5) for fixed c grows rapidly with c . Algorithms for computing λ_0 for fixed c can be found at [1, 9], [8, and the references therein]. There are no error estimates and no complexity analysis at the literature. W.H.J.Fuchs [3] proved the following asymptotic formula:

Theorem 3.2 (Fuchs Theorem). *Let $\lambda_0 \geq \lambda_1 \geq \lambda_2 \geq \lambda_3 \dots$ be the eigenvalues of the integral equation (5). Then*

$$(6) \quad 1 - \lambda_n \sim 4\pi^{1/2} 8^n (n!)^{-1} c^{n+1/2} e^{2c}$$

as $c \rightarrow \infty$.

□

We used H.Xiao, V.Rokhlin and N.Yarvin (XRY) [17] algorithm for computing λ_0 as a function of $c \in [0, 6]$ and compared it to the asymptotic formula (6) (see Figure 1). We used $k = 300$ for the complexity parameter in XRY [17] algorithm (see equation (54) and section 4 therein). The asymptotic formula is a good approximation for λ_0 starting from small c 's. The relative difference (i.e. (Numerical λ_0 - asymptotic λ_0)/(1- asymptotic λ_0)) is decreasing up to $c = 5$ and then it starts to increase. We believe that with higher complexity resources then us (We used PC) one should use numerical algorithms for $c \leq 10$. We believe that for $c > 10$ the asymptotic formula is good enough.

Possibility maps for different values of λ_0 are plotted in Figure 2. Below we will introduce possibility maps in term of spreading. This is an important difference between our approach to LPS approach.

LP [11] mentioned that their theorem can be used for describing the function $\phi(\alpha, \beta)$ that can be used for writing the uncertainty principle in the form:

$$(7) \quad c = \Omega T \geq \phi(\alpha, \beta)$$

or equivalently (to match Figure 2; LP used the concentration parameters α_T^2 and β_Ω^2 . We will discuss the parameters issue in section 5):

$$(8) \quad c = \Omega T \geq \Phi(\alpha^2, \beta^2) = \phi(\alpha, \beta)$$

To the inequalities 7 and 8 above and to their generalizations that will be introduced below we will call *Generalized Uncertainty Principles*.

LP used the concentrations $\alpha_T(f)$ (2) and $\beta_\Omega(f)$ (3) for formulating their generalized uncertainty principle. The notion of *spreading* is in some sense dual to the notion of *concentration*. In section 4 we will consider uncertainty principles for general weights using the spreading

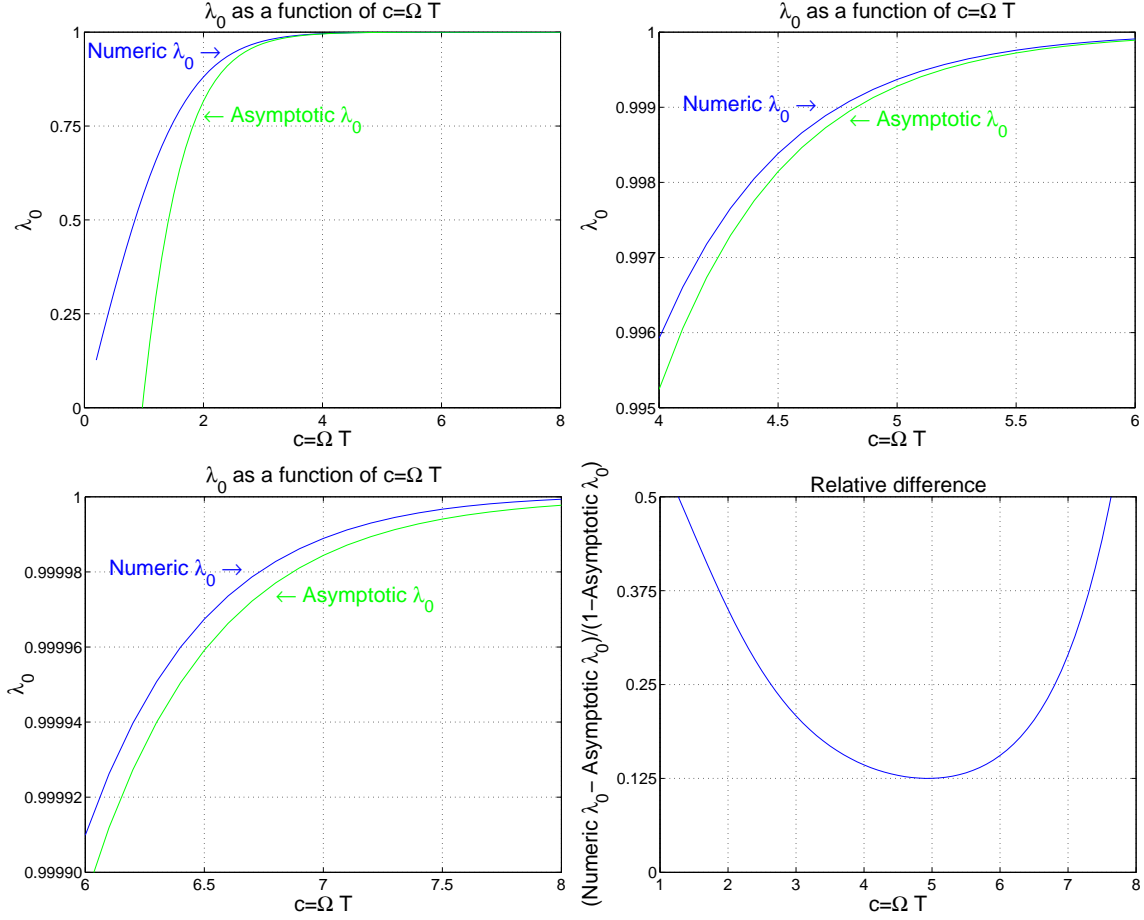


FIGURE 1. λ_0 as a function of $c = \Omega T$ and the relative difference between the asymptotic formula $(1 - 4\pi^{1/2}c^{1/2}e^{-2c})$ and the numerical calculations .

notion and describe the properties of the weights, $w_1(x)$ and $w_2(\omega)$ (see (11) and (12)), rigourously. In this section we define the time spreading (11) and frequency spreading (12) and use specific weights (9) and (10) that fits to LP Theorem. We will restate LP Theorem, using the spreadings $\gamma(f, a)$ and $\zeta(f, b)$ (see below (11), (12), (9) and (10)). Below we will redefine the terms "*possibility map*", "*possibility body*" and "*generalized uncertainty principle*" in an obvious way which fits the notion of spreading instead of the notion of concentration. We will use "the spreading language" from now on.

We define the time weight, $w_{1LP}(x)$, and the frequency weight, $w_{2LP}(\omega)$ as follows:

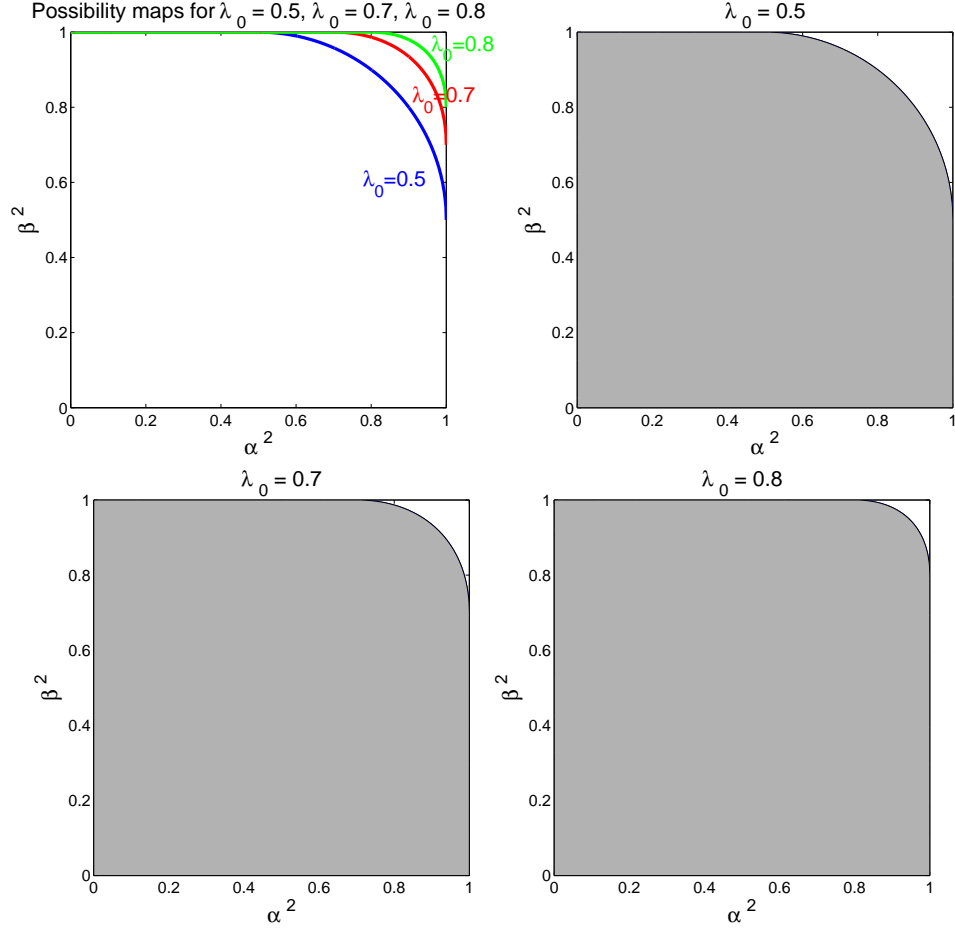


FIGURE 2. Possibility maps for $\lambda_0 = 0.5, \lambda_0 = 0.7, \lambda_0 = 0.8$ and the concentration parameters α^2 and β^2 .

$$(9) \quad w_{1LP}(x) = \begin{cases} 1 & \text{if } x \geq 1; \\ 0 & \text{else.} \end{cases}$$

$$(10) \quad w_{2LP}(\omega) = \begin{cases} 1 & \text{if } \omega \geq 1; \\ 0 & \text{else.} \end{cases}$$

We define the time spreading function as

Definition 3.3.

$$(11) \quad \gamma(f, a) = \frac{\left(\int_{-\infty}^{\infty} |f(x)|^2 S_a w_1(x) dx \right)^{\frac{1}{2}}}{\|f\|}$$

and the frequency spreading as

Definition 3.4.

$$(12) \quad \zeta(f, b) = \frac{\left(\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 S_b w_2(\omega) d\omega \right)^{\frac{1}{2}}}{\|f\|}$$

and see that in the case that $w_1(x) = w_{1LP}(x)$ and $w_2(\omega) = w_{2LP}(\omega)$ we have $\gamma^2(f, T) = 1 - \alpha_T^2(f)$ and $\zeta^2(f, \Omega) = 1 - \beta_\Omega^2(f)$.

Therefore, defining $\Psi(\gamma^2, \zeta^2)$ and $\psi(\gamma, \zeta)$ on $D_{LP^*} = [0, 1] \times [0, 1] \setminus \{(0, 0), (0, 1), (1, 0)\}$ by

$$\Psi(\gamma^2, \zeta^2) = \Phi(1 - \gamma^2, 1 - \zeta^2)$$

$$\psi(\gamma, \zeta) = \Psi(\gamma^2, \zeta^2)$$

we get the LP uncertainty principle in the "spreading language":

Theorem 3.5 (LP* Theorem). *There is a function f such that $\|f\| = 1$, $\gamma(f, T) = \gamma$ and $\zeta(f, \Omega) = \zeta$ under the following conditions and only under the following conditions:*

- 1) $\gamma = 0$ and $\sqrt{1 - \lambda_0} \leq \zeta < 1$
 - 2) $0 < \gamma \leq \sqrt{1 - \lambda_0}$ and $\cos^{-1} \sqrt{1 - \gamma^2} + \cos^{-1} \sqrt{1 - \zeta^2} \geq \cos^{-1} \sqrt{\lambda_0}$
 - 3) $\sqrt{1 - \lambda_0} < \gamma < 1$ and $0 \leq \zeta \leq 1$
 - 4) $\gamma = 1$ and $0 \leq \zeta \leq 1$
- where λ_0 is the largest eigenvalue of (5)

□

and transforming equation (8) to an equivalent general uncertainty principle in the "spreading language" we get:

$$(13) \quad c = \Omega T \geq \Psi(\gamma^2, \zeta^2) = \psi(\gamma, \zeta)$$

In section 4 we will see that the "spreading point of view" of LP theorem relates it to HPW uncertainty principle and to general uncertainty principles that are described there. We will redefine some of the terms that we used in the introduction, using the "spreading language"

We redefine the "possible area of level c ":

Definition 3.6. *The possible area of level c is the set:*

$$M_c = M_{\Omega, T} = \bigcup_{f \in L^2} (\gamma(f, T), \zeta(f, \Omega))$$

and redefine the "impossible area of level c ", the complement of $M_{\Omega, T}$ in D_{LP^*} :

Definition 3.7. *The impossible area of level c is the set:*

$$M'_c = D_{LP^*} \setminus M_c$$

and call the pair (M_c, M'_c) the possibility map of level c .

We redefine the "possibility body" by:

Definition 3.8. *The possibility body is the set:*

$$(14) \quad PB = \{(\alpha, \beta, c) | (\alpha, \beta) \in M_c, c \in (0, \infty)\}$$

Notice that the "possible areas" in Figure 2 (different λ_0 's corresponds to different c 's, see Figure 1) are convex. Analytic derivation of this fact is given below, in the example after Theorem 3.9. To uncertainty inequalities of the form (13), we will call "*Generalized Uncertainty Principle*" (we redefined our definition that follows (8) to fit the "spreading language").

ψ is a non-increasing function of γ for fixed ζ and a non-increasing function of ζ for fixed γ since $\lambda_0(c)$ is an non-decreasing function (see Figure 1)

Now we will derive classical uncertainty principles from the general uncertainty principle of LP.

Theorem 3.9. $\forall c = \Omega T > 0$ we have:

$$(15) \quad \sqrt{\int_{R \setminus [-T, T]} |f(x)|^2 dx} + \sqrt{\int_{R \setminus [-\Omega, \Omega]} |\hat{f}(\omega)|^2 d\omega} \geq \sqrt{1 - \lambda_0} \|f\|$$

Proof. We define $g(\gamma) = \cos^{-1}(\sqrt{1 - \gamma^2})$ on $[-1, 1]$. $g(\gamma)$ is symmetric with respect to the point 0 and on $[0, 1]$ we have: $g(\gamma) = \sin^{-1}(\gamma)$ on $[0, 1]$. therefore $g(\gamma)$ is concave.

ζ as a function of γ , $0 < \gamma \leq \sqrt{1 - \lambda_0}$ defined implicitly by $\cos^{-1} \sqrt{1 - \gamma^2} + \cos^{-1} \sqrt{1 - \zeta^2} = \cos^{-1} \sqrt{\lambda_0}$ is convex. To see that we write

$$g_2(\gamma, \zeta) = \cos^{-1} \sqrt{1 - \gamma^2} + \cos^{-1} \sqrt{1 - \zeta^2} = g(\gamma) + g(\zeta)$$

$g_2(\gamma, \zeta)$ is a concave function of two variables as a sum of two concave functions:

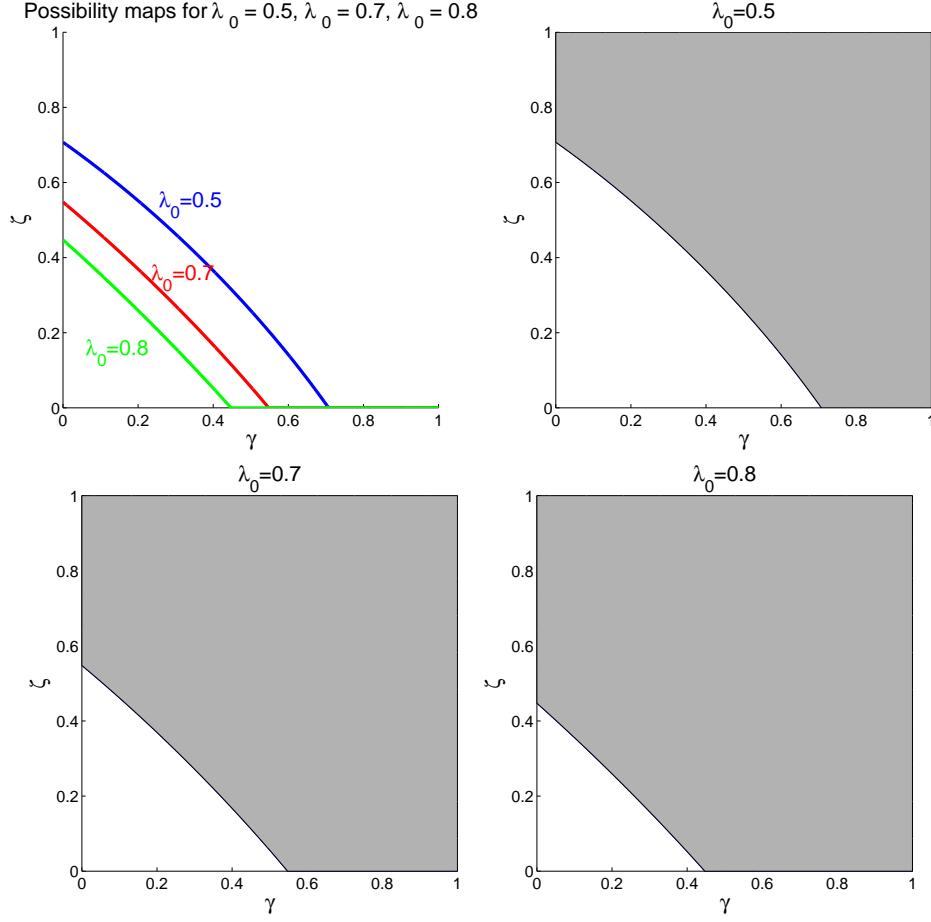


FIGURE 3. Possibility map for $\lambda_0 = 0.5, \lambda_0 = 0.7, \lambda_0 = 0.8$ using the coordinates (spreading parameters) γ and ζ .

$$\begin{aligned}
 g_2(\lambda(x_1, x_2) + (1 - \lambda)(x_2, y_2)) &= g_2((\lambda x_1 + (1 - \lambda)x_2, \lambda y_1 + (1 - \lambda)y_2)) \\
 &= g(\lambda x_1 + (1 - \lambda)x_2) + g(\lambda y_1 + (1 - \lambda)y_2) < \lambda g(x_1) + (1 - \lambda)g(x_2) + \\
 &\quad + \lambda g(y_1) + (1 - \lambda)g(y_2) = \lambda g_2(x_1, y_1) + (1 - \lambda)g_2(x_2, y_2)
 \end{aligned}$$

from the symmetry of $g(\gamma)$ it follows that $g_2(\gamma, \zeta)$ is symmetric with respect to the axes. Therefore ζ as a function of γ , $0 < \gamma \leq \sqrt{1 - \lambda_0}$ defined implicitly by $\cos^{-1} \sqrt{1 - \gamma^2} + \cos^{-1} \sqrt{1 - \zeta^2} = \cos^{-1} \sqrt{\lambda_0}$ is convex.

We have plotted ζ as a function of γ in Figure 3.

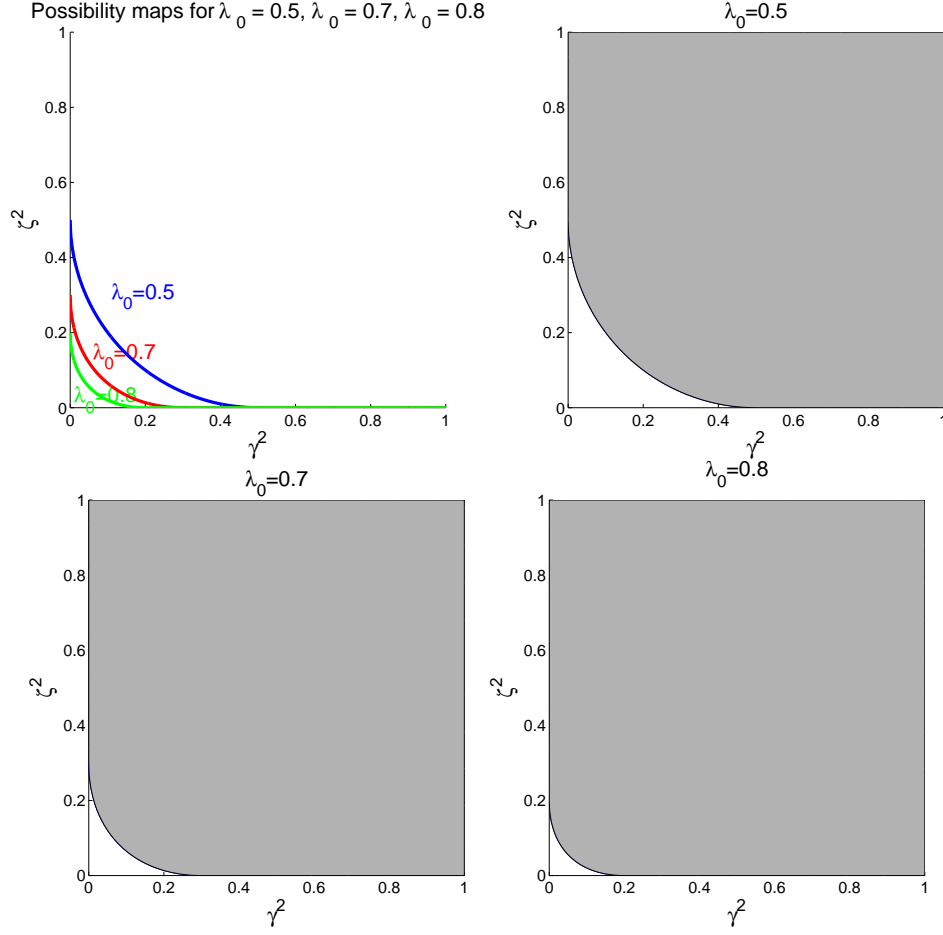


FIGURE 4. Possibility map for $\lambda_0 = 0.5, \lambda_0 = 0.7, \lambda_0 = 0.8$ using the coordinates (spreading parameters) γ^2 and ζ^2 .

thus the maximum of $\zeta + \gamma$ is at the two edge points $(\sqrt{1-\lambda_0}, 0)$, $(0, \sqrt{1-\lambda_0})$ and we have the following uncertainty principle:

$$(16) \quad \frac{\sqrt{\int_{R \setminus [-\frac{T}{2}, \frac{T}{2}]} |f(x)|^2 dx}}{\|f\|} + \frac{\sqrt{\int_{R \setminus [-\Omega, \Omega]} |\hat{f}(\omega)|^2 d\omega}}{\|f\|} \geq \sqrt{1-\lambda_0}$$

which implies (15).

□

In Theorem 3.9 above we showed that $M_c, c > 0$ is a non-convex set (we showed that ζ as a function of γ is convex, which is equivalent). We would like to remark that if we use a different set of natural parameters

(such as ζ^2 and γ^2 , see Figure 4) then M_c , where c belongs to some interval, may be a convex set.

As an example we show that using the parameters ζ^2 and γ^2 , M_c , where $\lambda_0(c) > \frac{1}{2}$, is a convex set: M_c is convex iff ζ^2 as a function of γ^2 , $0 < \gamma^2 \leq 1 - \lambda_0$ defined implicitly by $\cos^{-1} \sqrt{1 - \gamma^2} + \cos^{-1} \sqrt{1 - \zeta^2} = \cos^{-1} \sqrt{\lambda_0}$ is concave. To see that we write β^2 as a function of α^2

$$(17) \quad \beta^2(\alpha^2) = \cos^2(\cos^{-1} \sqrt{\lambda_0} - \cos^{-1} \sqrt{\alpha^2})$$

we denote $h(\alpha^2) = \cos^{-1} \sqrt{\lambda_0} - \cos^{-1} \sqrt{\alpha^2}$ and differentiate $\beta^2(\alpha^2)$ (β^2 with respect to α^2 twice) to get

$$(18) \quad (\beta^2)'(\alpha^2) = \frac{\cos(h(\alpha^2)) \sin(h(\alpha^2))}{\sqrt{\alpha^2(1 - \alpha^2)}}$$

$$(19) \quad (\beta^2)''(\alpha^2) = \frac{1}{2} \frac{\sin^2(h(\alpha^2))}{\sqrt{\alpha^2(1 - \alpha^2)}} - \frac{1}{2} \frac{\cos^2(h(\alpha^2))}{\sqrt{\alpha^2(1 - \alpha^2)}} +$$

$$(20) \quad + \frac{1}{2} \frac{\cos(h(\alpha^2)) \sin(h(\alpha^2))}{(\alpha^2)^{\frac{3}{2}}(1 - \alpha^2)^{\frac{1}{2}}} - \frac{1}{2} \frac{\cos(h(\alpha^2)) \sin(h(\alpha^2))}{(\alpha^2)^{\frac{1}{2}}(1 - \alpha^2)^{\frac{3}{2}}}$$

for $\lambda_0 \geq \frac{1}{2}$ since $\sin^2(h(\alpha^2)) \leq \cos^2(h(\alpha^2))$ and $(\alpha^2)^{\frac{3}{2}}(1 - \alpha^2)^{\frac{1}{2}} \geq (\alpha^2)^{\frac{1}{2}}(1 - \alpha^2)^{\frac{3}{2}}$ we get that $\beta^2(\alpha^2) \leq 0$. $\zeta^2(\gamma^2)$ is concave since $\beta^2(\alpha^2)$ is convex.

One can get different inequalities by minimizing different functions on the possible area. Of course, one can use different coordinates systems to work with for obtaining different inequalities. As an illustration we will use the coordinates ζ^2 and γ^2 . In this case we will not get a new uncertainty principle. We will get a weaker uncertainty principle then 15, but the illustration is instructive:

From symmetry with respect to the line $\zeta^2 = \gamma^2$ and concavity of the function $\zeta^2(\gamma^2)$, the maximum of the function $\gamma^2 + \zeta^2$ on the possibility map is attained at the point $\gamma^2 = \zeta^2 = 1 - \cos^2 \frac{\cos^{-1} \sqrt{\lambda_0}}{2}$. (We used the equation $2 \cos^{-1} \sqrt{1 - \gamma^2} = \cos^{-1} \sqrt{\lambda_0}$).

and we get:

$$(21) \quad \frac{\int_{-\infty}^{\infty} |f(x)|^2 S_T w_{1LP}(x) dx}{\|f\|^2} + \frac{\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 S_{\Omega} w_{2LP}(\omega) d\omega}{\|f\|^2} \geq 2(1 - \cos^2 \frac{\cos^{-1} \sqrt{\lambda_0}}{2})$$

using the identity:

$$\cos \frac{u}{2} = \sqrt{\frac{1 + \cos u}{2}}$$

we get the uncertainty principle:

$$(22) \quad \sqrt{\int_{R \setminus [-T, T]} |f(x)|^2 dx + \int_{R \setminus [-\Omega, \Omega]} |\hat{f}(\omega)|^2 d\omega} \geq \sqrt{1 - \lambda_0} \|f\|$$

As we mentioned the uncertainty principle (22) is weaker than (15). In fact it is a consequence of (15) using the inequality

$$\sqrt{2}\sqrt{a+b} \geq \sqrt{a} + \sqrt{b}$$

Remark: We wanted to point out the convexity of $\zeta(\gamma)$ and the concavity of $\zeta^2(\gamma^2)$. If one is only interested in finding the inequalities, there may be easier ways to get those.

4. UNCERTAINTY PRINCIPLES FOR GENERAL WEIGHTS

In this section we will generalize the example from section 3 to more general weights. We denote the time weight by $w_1(x)$ and the frequency weight by $w_2(\omega)$. We will use the notations:

$$w_{1a}(x) = S_a w_1(x) = w_1(xa^{-1})$$

where a is called the time weight scaling parameter; and

$$w_{2b}(\omega) = S_b w_2(\omega) = w_2(\omega b^{-1})$$

where b is called the frequency weight scaling parameter. We recall the definitions of time spreading and frequency spreading (see definitions 3.3 and 3.4:

The time spreading:

$$(23) \quad \gamma(f, a) = \frac{\left(\int_{-\infty}^{\infty} w_{1a}(x) |f(x)|^2 dx \right)^{\frac{1}{2}}}{\|f\|}$$

The frequency spreading:

$$(24) \quad \zeta(f, b) = \frac{\left(\int_{-\infty}^{\infty} w_{2b}(\omega) |\hat{f}(\omega)|^2 d\omega \right)^{\frac{1}{2}}}{\|f\|}$$

In section 3 we used specific weights $w_{1LP}(x)$ (see (9)) and $w_{2LP}(\omega)$ (see (10)). In this section we will use general weights that posses mild requirements.

Definition 4.1. A point $(p, q, a, b) \in R^{4+}$ is called *realizable* iff there is a function $f \in L_2$ such that: $p = \gamma(f, a)$, $q = \zeta(f, b)$, $a > 0, b > 0$

Lemma 4.2. A point $(p, q, a_1, b_1) \in R^{4+}$ is *realizable* iff the point $(p, q, ka_1, \frac{b_1}{k}), k > 0$ is *realizable*

Proof. It is enough to show that If

$$\|f\| = 1, p = \gamma(f, a_1), q = \zeta(f, b_1), a_2 = ka_1, b_2 = \frac{b_1}{k}$$

then the function $f_k(x) = \frac{1}{\sqrt{k}}f(\frac{x}{k})$ satisfies:

$$\|f_k\| = 1, p = \gamma(f_k, a_2), q = \zeta(f_k, b_2)$$

$$\begin{aligned} \gamma(f_k, a_2) &= \left(\int_{-\infty}^{\infty} w_1\left(\frac{x}{a_2}\right) \left| \frac{1}{\sqrt{k}} f\left(\frac{x}{k}\right) \right|^2 dx \right)^{\frac{1}{2}} \\ &= \left(\int_{-\infty}^{\infty} w_1\left(\frac{y}{a_1}\right) \frac{1}{k} |f(y)|^2 k dy \right)^{\frac{1}{2}} \\ &= \gamma(f, a_1) = p \end{aligned}$$

where $y = \frac{x}{k}$

We use $\frac{1}{\sqrt{k}}f(\frac{x}{k}) = \sqrt{k}\hat{f}(k\omega)$ and see:

$$\begin{aligned} \zeta(f_k, b_2) &= \left(\int_{-\infty}^{\infty} w_2\left(\frac{\omega}{b_2}\right) |\sqrt{k}\hat{f}(k\omega)|^2 d\omega \right)^{\frac{1}{2}} \\ &= \left(\int_{-\infty}^{\infty} w_2\left(\frac{v}{b_1}\right) k |\hat{f}(v)|^2 \frac{dv}{k} \right)^{\frac{1}{2}} \\ &= \zeta(f, a_1) = q \end{aligned}$$

where $v = k\omega$

□

Note that Lemma 4.2 is steal correct if we use a change of coordinates from R^{4+} to R^{4+} of the form $(p, q, a, b) \rightarrow (s(p, q), t(p, q), a, b)$. We will use specific change of variables in Section 5.

Lemma 4.2 indicates that the relevant parameter is the time weight scaling parameter \times the frequency weight scaling parameter; So, we can define realizable points in R^{3+} instead of in R^{4+} :

Definition 4.3. A point $(p, q, c) \in R^{3+}$ is called *realizable* iff there is a function $f \in L_2$ such that, $\|f\| = 1$, $p = \gamma(f, a)$, $q = \zeta(f, b)$ and $c = ab$.

Definition 4.4. We will call the set of realizable points in R^{3+} "possibility body" and denote it by PB_{γ^1, ζ^1} .

Note that we may think of the set PB_{γ^1, ζ^1} as the set:

$$\bigcup_{a, b > 0, f \in L_2} (\gamma(f, a), \zeta(f, b), a, b)$$

where we identify points such that $ab = c$.

The meaning of the subindexes will be clear in Section 5, since definitions 4.1, 4.3 and 4.4 are special cases of the more general definitions 5.1, 5.2 and 5.3 in section 5. In the following it also will become clear that the possibility body PB defined in Definition 3.8. is similar to PB_{γ^1, ζ^1} when we use LP-weights (9) and (10).

We will use the following type of weights in the theorem. The weights in LP result , (9) and (10), are pointwise limit of weights of type 1.

Definition 4.5 (Weight of type 1). A weight $w(x)$ is of type 1 if and only if:

- a) $w(0) = 0$.
- b) $w(x)$ is a continuous function.
- c) $w(x)$ is an even function i.e. $w(x) = w(-x)$.
- d) $w(x)$ is strictly increasing on $[0, \infty]$ i.e. $\forall 0 \leq x < y$ $w(x) < w(y)$.
- e) $w(x)$ tends to a finite number as x tends to ∞ i.e.
 $\lim_{x \rightarrow \infty} w(x) = L < \infty$

Restricting ourselves to weights of type 1 we get the following generalized uncertainty principle:

Theorem 4.6 (General Uncertainty Theorem For Weights Of Type 1). Let $w_1(x)$ and $w_2(\omega)$ be weights of type 1 where

$$\lim_{x \rightarrow \infty} w_1(x) = P$$

and

$$\lim_{x \rightarrow \infty} w_2(x) = Q$$

Then the possibility body, PB_{γ^1, ζ^1} , is defined, up to a set of measure zero, by a generalized uncertainty inequality of the form:

$$(25) \quad ab = c \geq \psi_{w_1(x), w_2(\omega)}(\gamma, \zeta)$$

where ψ is defined on the open square $N = (0, \sqrt{P}) \times (0, \sqrt{Q})$

$\psi_{w_1(x), w_2(\omega)}(\gamma, \zeta)$ is a non-increasing function of γ for fixed ζ and a non-increasing function of ζ for fixed γ

If $w_1(x) = w_2(x)$ then $\psi_{w_1(x), w_2(\omega)}(\gamma, \zeta)$ is symmetric

Proof. The proof has 4 steps:

Step 1:

If a point (c, γ_1, ζ) is realizable then all points (c, γ_2, ζ) , $\gamma_1 < \gamma_2 < \sqrt{P}$, are realizable.

To see that, we take a function f such that $\|f\| = 1$, $\gamma_1 = \gamma(f, a)$, $\zeta = \zeta(f, b)$ where $c = ab$;

The spreading in time of the translation of f ,

$$\gamma(f, a, x_0) = \int_{-\infty}^{\infty} |f(x - x_0)|^2 w_{1a}(x) dx$$

is a continuous function of x_0 . $\gamma(f, a, 0) = \gamma_1$ and

$$\lim_{x_0 \rightarrow \infty} \gamma(f, a, x_0) = \sqrt{P}$$

So for some x_0 $\gamma(f, a, x_0) = \gamma_2$.

The spreading in frequency of the translation of f is a constant,

$$\zeta = \frac{\left(\int_{-\infty}^{\infty} w_2(\omega) |\hat{f}(\omega)|^2 \right)^{\frac{1}{2}}}{\|f\|}$$

and therefore (c, γ_2, ζ) is realizable.

In the same way: If a point (c, γ, ζ_1) is realizable then all points (c, γ, ζ_2) , $\zeta_1 < \zeta_2 < \sqrt{Q}$, are realizable.

Step 2:

$\forall(\gamma, \zeta)$ in the square N , $\exists c > 0$ such that (c, γ, ζ) is realizable:

To see that, we take an arbitrary f with norm $\|f\| = 1$. The spreading functions $\gamma(f, a)$, $\zeta(f, b)$ are defined and continuous on the interval $(0, \infty)$ as functions of a and b respectively. $\lim_{a \rightarrow \infty} \gamma(f, a) = 0$, $\lim_{b \rightarrow \infty} \zeta(f, b) = 0$ and therefore $\exists \gamma_1 < \gamma$, $\zeta_1 < \zeta$, $c = ab$ such that the point (c, γ_1, ζ_1) is realizable and by the first part of the proof (c, γ, ζ) is realizable.

Definition 4.7. $\forall(\gamma, \zeta) \in N$ the function $\psi(\gamma, \zeta)$ is defined as:

$$(26) \quad \psi(\gamma, \zeta) = \inf\{c \mid \text{The point } (c, \gamma, \zeta) \text{ is realizable}\}$$

Note that the points of $\partial N \times [0, \infty)$ are not realizable, because of the properties of the weights.

Step 3:

The function $\psi_{w_1(x), w_2(\omega)}(\gamma, \zeta)$ is a non-increasing function of γ for fixed ζ since from what we have shown above we get that for $\gamma_1 < \gamma_2$ we have the following inclusion

$$(27) \quad \{c \mid \text{The point } (c, \gamma_1, \zeta) \text{ is realizable}\} \subseteq \{c \mid \text{The point } (c, \gamma_2, \zeta) \text{ is realizable}\}$$

In the same way we get that $\psi(\gamma, \zeta)$ is a non-increasing function of ζ for fixed γ .

The case $w_1(x) = w_2(x)$: If a point (c, γ_1, ζ_1) is realizable, then there is a function such that $\|f\| = 1$, $\gamma(f, a) = \gamma_1$, $\zeta(f, b) = \zeta_1$ and $c = ab$. Since $\hat{f}(x) = f(-x)$, we have for \hat{f} , $\|\hat{f}\| = 1$, $\gamma(\hat{f}, b) = \zeta_1$, $\zeta(\hat{f}, a) = \gamma_1$ and $c = ba$ which means that the point (c, ζ_1, γ_1) is realizable and therefore $\psi(\gamma, \zeta)$ is symmetric.

Step 4:

If a point (c_1, γ_0, ζ_0) is realizable then the points (c_2, γ_0, ζ_0) , $c_2 \geq c_1$ are realizable: Without loss of generality we can take a_1, b_1 s.t. $a_1 b_1 = c_1$ and b_2 s.t. $a_1 b_2 = c_2$. If $\exists f$ s.t. $\gamma(f, a_1) = \gamma_0$, $\zeta(f, b_1) = \zeta_0$ then for this f $\zeta(f, b_2) = \zeta_1 < \zeta_0$ since

$$\left(\int_{-\infty}^{\infty} w_{2b_1}(\omega) |\hat{f}(\omega)|^2 \right)^{\frac{1}{2}} \|f\|^{-1} > \left(\int_{-\infty}^{\infty} w_{2b_2}(\omega) |\hat{f}(\omega)|^2 \right)^{\frac{1}{2}} \|f\|^{-1}$$

from the monotonicity of $w_2(\omega)$.

This means that the point (γ_0, ζ_1, c_2) is realizable and from step 1 it follows that (γ_0, ζ_0, c_2) is realizable.

From our construction the set of points that fulfill inequality (25) and the possibility body are equal up to the graph of $\psi_{w_1(x), w_2(\omega)}(\gamma, \zeta)$, which is a set of measure zero.

□

We will prove now a similar theorem for weights $w(x)$ s.t. $\lim_{x \rightarrow \infty} w(x) = \infty$. The structure of the proof is the same. We will state the theorem and explain the necessary modifications in the proof.

We will define the type of weights we will have in the theorem. The weight in the HPW case, $w(x) = x^2$, is an example of a weight of this type. Property 1 of the weights will use the function $C(h, x)$ which is related to the weight as follows:

Definition 4.8.

$$C(h, x_*) = \sup\left\{\frac{w(x+h)}{w(x)} \mid x \geq x_*\right\}.$$

$$\forall(h, x_*) \in (0, \infty) \times (0, \infty)$$

Note that $C(h, x)$ is a non-increasing function of x for fixed h . Note that if $w(x)$ is a non-decreasing function on $[0, \infty)$ then $C(h, x)$ is a non-decreasing function of h for fixed x since $h_2 \geq h_1 > 0$ implies $\frac{w(x+h_2)}{w(x)} \geq \frac{w(x+h_1)}{w(x)}$.

Definition 4.9 (Property 1). *A weight has property 1 iff, $\forall h_0 > 0 \exists x_0 > 0$ such that $C(h_0, x_0)$ is finite.*

Definition 4.10 (Weight of type ∞). *A weight $w(x)$ is of type ∞ if and only if:*

- a) $w(0) = 0$.
- b) $w(x)$ is a continuous function.
- c) $w(x)$ is an even function i.e. $w(x) = w(-x)$.
- d) $w(x)$ is strictly increasing on $[0, \infty]$ i.e. $\forall 0 \leq x < y \ w(x) < w(y)$.
- e) $w(x)$ tends to ∞ as x tends to ∞ not faster than some polynomial, i.e. $\lim_{x \rightarrow \infty} w(x) = \infty$, $w_1(x) \leq P(x)$ where $P(x)$ is a polynomial.
- f) $w(x)$ obtains property 1.

Theorem 4.11 (General Uncertainty Theorem For Weights Of Type ∞). *Let $w_1(x)$ and $w_2(x)$ be weights of type ∞*

Then the possibility body, PB_{γ^1, ζ^1} , is defined, up to a set of measure zero, by a generalized uncertainty inequality of the form:

$$(28) \quad ab = c \geq \psi_{w_1(x), w_2(\omega)}(\gamma, \zeta)$$

where ψ is defined on the open upper-right quarter of the plain $N = (0, \infty) \times (0, \infty)$.

$\psi(\gamma, \zeta)$ is a non-increasing function of γ for fixed ζ and a non-increasing function of ζ for fixed γ .

If $w_1(x) = w_2(x)$ then $\psi_{w_1(x), w_2(\omega)}(\gamma, \zeta)$ is symmetric.

Proof. In Theorem 4.6, in step 1, the continuity of

$\gamma(f, a, x_0) = \int_{-\infty}^{\infty} |f(x - x_0)|^2 w_{1a}(x) dx$ in x_0 was obvious. Here Some elaboration is needed. We will proof continuity from the right (continuity from the left can be done in the same way). We will live it as an exercise to show that if $\gamma(f, a, x_0)$ exists then $\gamma(f, a, x_1)$ exists for all $x_1 > x_0$. Note that it is enough to show continuity at $\gamma(f, a, 0)$.

So we show first continuity at $x_0 = 0$ of $\gamma(f, a, x_0)$. We fix an arbitrary positive number $h_0 > 0$, and choose $q_1 < 0$ such that $\int_{-\infty}^{q_1} f(x)w(x)dx < \epsilon_1$. From property 1, $\exists x_p > h_0$ s.t. $C(h_0, x_p)$ is finite. We choose $q_2 > x_p$ such that $\int_{q_2}^{\infty} |f(x)|^2 w(x)dx < \epsilon_3$

We have:

$$\begin{aligned} \int_{-\infty}^{\infty} (|f(x-h)|^2 - |f(x)|^2)w(x)dx &= \int_{-\infty}^{q_1} (|f(x-h)|^2 - |f(x)|^2)w(x)dx + \\ &+ \int_{q_1}^{q_2+h_0} (|f(x-h)|^2 - |f(x)|^2)w(x)dx \\ (29) \quad &+ \int_{q_2+h_0}^{\infty} (|f(x-h)|^2 - |f(x)|^2)w(x)dx \end{aligned}$$

We will check the three terms separately:

The first term of (31):

$$\forall 0 < h < |q_1|$$

$$\begin{aligned} (30) \quad \int_{-\infty}^{q_1} |f(x-h)|^2 w(x)dx &= \int_{-\infty}^{q_1-h} |f(y)|^2 w(y+h)dy \leq \int_{-\infty}^{q_1} |f(y)|^2 w(y+h)dy \leq \\ &\leq \int_{-\infty}^{q_1} |f(y)|^2 w(y)dy < \epsilon_1 \end{aligned}$$

and from the triangle inequality we have:

$$\int_{-\infty}^{q_1} (|f(x-h)|^2 - |f(x)|^2)w(x)dx < \int_{-\infty}^{q_1} (||f(x-h)|^2 - |f(x)|^2|)w(x)dx < 2\epsilon_1$$

The second term of (31):

Definition 4.12. $D = \max\{w(q_1), w(q_2 + h_0)\}$.

$$\forall \epsilon_2 > 0, \exists 0 < h_1 < \min\{|q_1|, h_0\} \text{ s.t.}$$

$$\int_{q_1}^{q_2+h_0} (|f(x-h_1)|^2 - |f(x)|^2)w(x)dx < D \int_{-\infty}^{\infty} (||f(x-h_1)|^2 - |f(x)|^2|)dx < \epsilon_2$$

since $|f(x)|^2 \in L_1$ and $\forall g(x) \in L_1$ we have $\lim_{h \rightarrow 0} \int_{-\infty}^{\infty} |g(x-h) - g(x)| = 0$

The third term of (31):

$$\begin{aligned} \int_{q_2+h_0}^{\infty} |f(x-h_1)|^2 w(x)dx &= \int_{q_2+h_0-h_1}^{\infty} |f(y)|^2 w(y) \frac{w(y+h_1)}{w(y)} dy < \\ &< C(h_0, x_0) \epsilon_3 \end{aligned}$$

where we used the monotonicity of $C(h, x)$. By the triangle inequality we have:

$$\begin{aligned} &\int_{q_2+h_0}^{\infty} (|f(x-h_1)|^2 - |f(x)|^2)w(x)dx \leq \\ &\leq \int_{q_2+h_0}^{\infty} (||f(x-h_1)|^2 - |f(x)|^2|)w(x)dx \leq \\ (31) \quad &\leq (1 + C(h_0, x_0))\epsilon_3 \end{aligned}$$

Taking $\epsilon_1 = \frac{\epsilon}{3}$, $\epsilon_2 = \frac{\epsilon}{3}$ and $\epsilon_3 = \frac{1}{3(1+C(h_0, x_p))}$ and h_1 as above we get the continuity of $\gamma(f, a, x_0)$ as a function of x_0 at $x_0 = 0$.

In step 2 the only modification we need is that instead of taking an arbitrary $f(x)$ we take $f(x) = \exp(-x^2)$.

No modification in steps three and four is needed.

□

Note that in Theorem 4.6 $N = (0, \sqrt{P}) \times (0, \sqrt{Q})$ and in Theorem 4.11 $N = (0, \infty) \times (0, \infty)$.

Now we define similar definitions as in the LPS case (Definitions 3.6 and 3.7) for weights of type 0 and weights of type ∞ :

Definition 4.13. *The possibility area of level c is the set: $M_c = \{(p, q) | (p, q, c) \text{ is realizable}\}$*

Definition 4.14. *The impossible area of level c is the set $M'_c = N \setminus M_c$*

Note that we have already defined the possibility body at Definition 4.4.

To the pair (M_c, M'_c) we will call "The possibility map of level c ".

4.1. The Heisenberg-Pauli-Weyl General Uncertainty Principle. It is easy to see that the HPW weights i.e. $w_1(x) = w_2(x) = x^2$ are of type ∞ . We use the HPW uncertainty principle to compute the function $\psi(\gamma, \zeta)$ explicitly and then we find the boundary of the possible area of level c (see Figure 5).

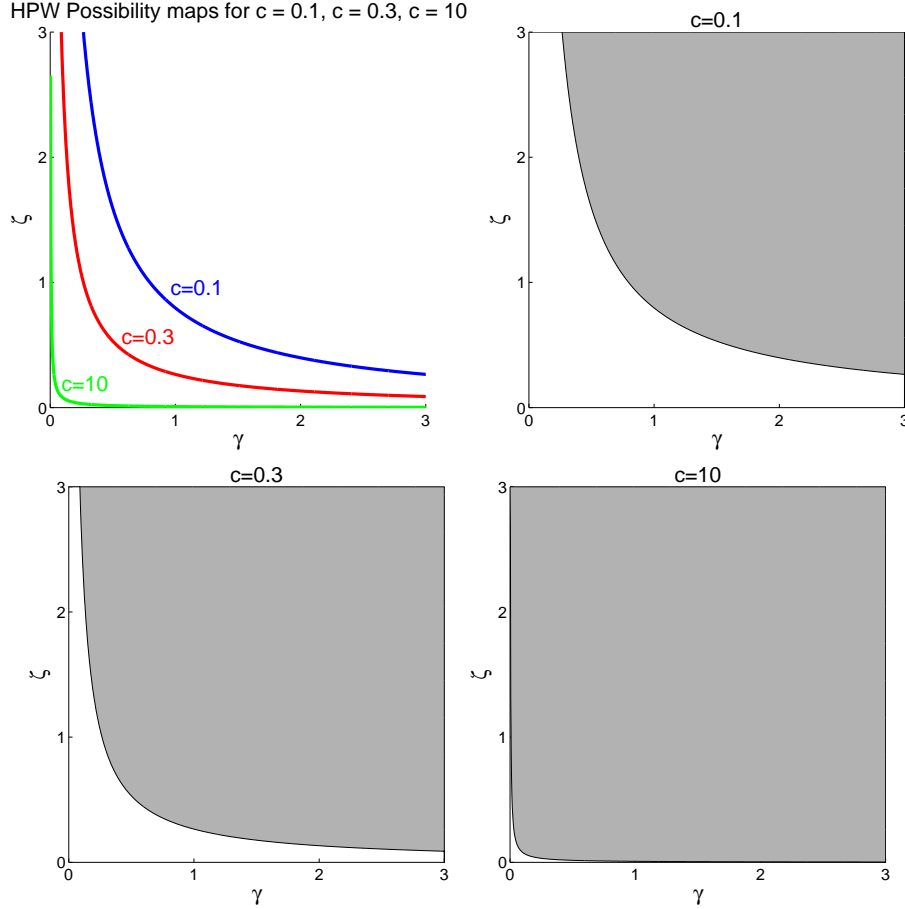


FIGURE 5. Heisenberg-Pauli-Weyl Possibility maps for $c = 0.1, c = 0.3, c = 10$ using the coordinates (spreading parameters) ζ and γ .

from:

$$(32) \quad \int_{-\infty}^{\infty} \left(\frac{x}{a}\right)^2 |f(x)|^2 dx \int_{-\infty}^{\infty} \left(\frac{\omega}{b}\right)^2 |\hat{f}(\omega)|^2 d\omega = \left(\frac{1}{ab}\right)^2 \int_{-\infty}^{\infty} x^2 |f(x)|^2 dx \int_{-\infty}^{\infty} \omega^2 |\hat{f}(\omega)|^2 d\omega$$

we write

$$a^2\gamma^2(f, a)b^2\zeta^2(f, b) = \gamma^2(f, 1)\zeta^2(f, 1) \geq \left(\frac{1}{4\pi}\right)^2$$

or

$$a^2b^2 \geq \left(\frac{1}{4\pi}\right)^2 \frac{1}{\gamma^2(f, a)\zeta^2(f, b)}$$

we define:

$$(33) \quad \theta(\gamma, \zeta) = \frac{1}{4\pi} \frac{1}{\gamma\zeta}$$

and get:

$$c = ab \geq \theta(\gamma, \zeta)$$

We define:

$$(34) \quad u_d(x) = \sqrt{\frac{d}{\sqrt{2}}} e^{-\pi(\frac{x}{d})^2}$$

We live it as an exercise to check that $\|u_d(x)\| = 1$, $\gamma(u_d(x), a)$ is continuous as a function of d , $\lim_{d \rightarrow 0} \gamma(u_d(x), a) = 0$ and $\lim_{d \rightarrow \infty} \gamma(u_d(x), a) = \infty$.

from HPW Theorem and (32) it follows that $\forall a, b, d > 0$:

$$(35) \quad \gamma(u_d(x), a)\zeta(u_d(x), b)ba = \frac{1}{4\pi}$$

and therefore the graph of $\theta(\gamma, \zeta)$ is equal:

$$(36) \quad \bigcup_{d \in (0, \infty), c \in (0, \infty)} (\gamma(u_d(x), a), \zeta(u_d(x), b), c)$$

(where again the dependence on a and b separately is not important), which means that the infimum in (26) is attained and

$$(37) \quad \psi(\gamma, \zeta) = \theta(\gamma, \zeta)$$

□

From (33) and (37) we see that the boundary of the possible area of level c in the HPW case, when we use the spreading parameters is (see Figure 5):

$$(38) \quad \gamma = \frac{1}{4\pi c\zeta}$$

5. NATURAL COORDINATES SYSTEMS FOR THE POSSIBILITY BODY AND CONVEXITY

As we saw the notion of generalized uncertainty principles has different settings. In LP Theorem (Theorem 3.1) we use concentration parameters and the function $\phi(\alpha, \beta)$ is defined on D_{LP} . In LP* Theorem (Theorem 3.5) we use spreading parameters and the function $\psi(\gamma, \zeta)$ is defined on D_{LP^*} . In General Uncertainty Theorem For Weights Of Type 0 (Theorem 4.6) and General Uncertainty Theorem For Weights Of Type ∞ (Theorem 4.11) we use spreading parameters and $\psi(\gamma, \zeta)$ is defined on $(0, \sqrt{P}) \times (0, \sqrt{Q})$ and $(0, \infty) \times (0, \infty)$ respectively. The different settings of the "General uncertainty principles" indicates that *there is a new phenomenon underline those*.

We saw that we can use different coordinates for representing the generalized uncertainty principles (see 8 and 13). Below we see that it is equivalent to measuring concentration (spreading) in different ways. In this section we discuss the question of the existence of natural coordinates for describing the phenomenon of "generalized uncertainty principles" and the question of the convexity of the possibility body.

We start by showing that the boundary of the possible area in the LP case is an algebraic curve, when we use the concentration parameters α and β (see (2), (3) and LP Theorem - Theorem 3.1) or the spreading parameters γ and ζ (see definitions 3.3 and 3.4 and LP* Theorem - Theorem 3.5)

Using the concentration parameters we have:

$$\begin{aligned} \cos^{-1} \alpha + \cos^{-1} \beta &= \cos^{-1} \sqrt{\lambda_0} \\ \alpha\beta - \sqrt{1-\alpha^2}\sqrt{1-\beta^2} &= \sqrt{\lambda_0} \\ \alpha^2\beta^2 - 2\sqrt{\lambda_0}\alpha\beta + \lambda_0 &= (1-\alpha^2)(1-\beta^2) \\ \alpha^2 + \beta^2 - 2\sqrt{\lambda_0}\alpha\beta &= 1 - \lambda_0 \\ \frac{1}{1-\lambda_0}\alpha^2 + \frac{1}{1-\lambda_0}\beta^2 - \frac{\sqrt{\lambda_0}}{1-\lambda_0}\alpha\beta &= 1 \end{aligned}$$

$$(39) \quad \begin{pmatrix} \alpha & \beta \end{pmatrix} \frac{1}{1-\lambda_0} \begin{pmatrix} 1 & -\sqrt{\lambda_0} \\ -\sqrt{\lambda_0} & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 1$$

$$\begin{pmatrix} \alpha & \beta \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \frac{1}{1-\lambda_0} \begin{pmatrix} 1-\sqrt{\lambda_0} & 0 \\ 0 & 1+\sqrt{\lambda_0} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 1$$

and we see that using the coordinates:

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

we have the ellipse in simple form:

$$(40) \quad \frac{u^2}{\left(\sqrt{\frac{1-\lambda_0}{1-\sqrt{\lambda_0}}}\right)^2} + \frac{v^2}{\left(\sqrt{\frac{1-\lambda_0}{1+\sqrt{\lambda_0}}}\right)^2} = 1$$

Thus using the concentration parameters α and β the boundary of the possibility area consists of straight lines and part of an ellipse (see Figure 6) which its main axis is in the direction

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

and its minor axis is in the direction

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

We calculate the distance of the ellipse focuses from the origin

$$(41) \quad c = \left(\left(\sqrt{\frac{1-\lambda_0}{1-\sqrt{\lambda_0}}} \right)^2 - \left(\sqrt{\frac{1-\lambda_0}{1+\sqrt{\lambda_0}}} \right)^2 \right)^{\frac{1}{2}} = 2^{\frac{1}{2}} \lambda_0^{\frac{1}{4}}$$

and see that the focuses of the ellipse are placed at

$$2^{\frac{1}{2}} \lambda_0^{\frac{1}{4}} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

and

$$-2^{\frac{1}{2}} \lambda_0^{\frac{1}{4}} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

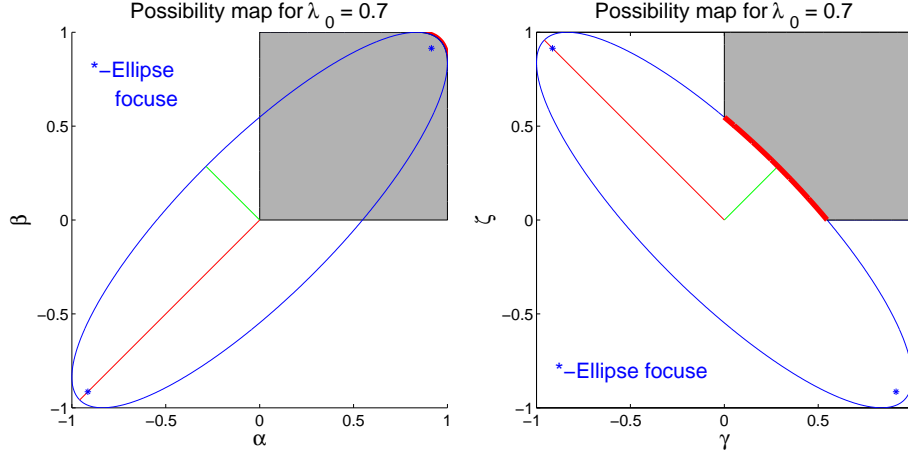


FIGURE 6. Possibility maps and the related ellipses for $\lambda_0 = 0.7$, using the coordinates (concentration parameters) α and β on the left and using the coordinates (spreading parameters) ζ and γ on the right.

Now we show that the boundary of the possibility area consists of straight lines and part of an ellipse (see Figure 6) when we use the spreading parameters γ and ζ .

Similar calculations to the concentration case above gives:

$$\cos^{-1} \sqrt{1 - \gamma^2} + \cos^{-1} \sqrt{1 - \zeta^2} = \cos^{-1} \sqrt{\lambda_0}$$

$$(\gamma \ \zeta) \frac{1}{1 - \lambda_0} \begin{pmatrix} 1 & \sqrt{\lambda_0} \\ \sqrt{\lambda_0} & 1 \end{pmatrix} \begin{pmatrix} \gamma \\ \zeta \end{pmatrix} = 1$$

$$(\gamma \ \zeta) \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \frac{1}{1 - \lambda_0} \begin{pmatrix} 1 + \sqrt{\lambda_0} & 0 \\ 0 & 1 - \sqrt{\lambda_0} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \gamma \\ \zeta \end{pmatrix} = 1$$

In this case the main axis of the ellipse is in the direction

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

and its minor axis is in the direction

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

The focuses of the ellipse are at the points

$$2^{\frac{1}{2}}\lambda_0^{\frac{1}{4}}\left(\frac{\frac{1}{\sqrt{2}}}{-\frac{1}{\sqrt{2}}}\right)$$

and

$$-2^{\frac{1}{2}}\lambda_0^{\frac{1}{4}}\left(\frac{\frac{1}{\sqrt{2}}}{-\frac{1}{\sqrt{2}}}\right)$$

In the HPW case (see (33) and (37)), when we use γ^m , $m \in N$ and ζ^n , $n \in N$ as parameters (we use the notation $\Psi(\gamma^m, \zeta^n)$) we get:

$$(42) \quad \Psi(\gamma^m, \zeta^n) = \psi(\gamma, \zeta)$$

i.e

$$(43) \quad \Psi(\gamma^m, \zeta^n) = \frac{1}{4\pi} \frac{1}{\sqrt[m]{\gamma^m} \sqrt[n]{\zeta^n}}$$

We see that the boundary of the possibility area of level c is again an algebraic curve defined by:

$$(44) \quad (\gamma^m)^n (\zeta^n)^m = \frac{1}{(4\pi c)^{mn}}$$

The question of finding types of weights and coordinate systems such that the boundary of the possible areas of different levels (different c 's) are algebraic curves may indicate what are the natural coordinate systems for describing the general uncertainty principles phenomenon.

As we saw in section 3 the question of convexity of the possibility body depends on the set of parameters we choose for describing the possibility body. (In section 3 we had a convex possible area for the parameter set (γ^2, ζ^2, c) and a non-convex possible area for the parameter set (γ, ζ, c) .) The following definitions (Definitions 5.1, 5.2 and 5.3) relates the coordinate systems to different ways of measuring the time and frequency spreadings. Motivated by the HPW case we generalize definitions 4.1, 4.3 and 4.4 as follows:

Definition 5.1. A point $(p, q, a, b) \in R^{4+}$ is called realizable with respect to γ^m , and ζ^m , $m \in N$ iff there is a function $f \in L_2$ such that: $p = \gamma^m(f, a)$ and $q = \zeta^m(f, b)$, $a > 0, b > 0$

Note that one can think about the symbol γ^m (respectively ζ^m) as the function γ (respectively ζ) to the power m or as a new function for measuring spreading. In 42 we use the symbols γ^m and ζ^m also as a spreading coordinates. Bellow we will continue to use γ^m and ζ^n with

their different meanings. The meaning will be clear from the context. Using Lemma 4.2 and the note that follows it we can define:

Definition 5.2. A point $(p, q, c) \in R^{3+}$ is called *realizable with respect to γ^m and ζ^m* , $m \in N$ iff there is a function $f \in L_2$ such that, $\|f\| = 1$, $p = \gamma^m(f, a)$, $q = \zeta^m(f, b)$ and $c = ab$.

Definition 5.3. We will call the set of realizable points with respect to γ^m , and ζ^m , $m \in N$, in R^{3+} "possibility body of order m " and denote it by PB_{γ^m, ζ^m} .

It is easy to see that the following relation holds:

$$(45) \quad (p^m, q^m, c) \in PB_{\gamma^m, \zeta^m} \text{ iff } (p, q, c) \in PB_{\gamma^1, \zeta^1}$$

In the following we will continue to discuss the case of weights of type ∞ .

Lemma 5.4. For every fixed $\gamma_1 > 0$ and $c_1 > 0$, there exists a point (γ_1, ζ, c_1) that is realizable.

For every fixed ζ_1 and c_1 , there exists a point (γ, ζ_1, c_1) that is realizable.

Proof. We fix $a, b > 0$ such that $ab = c_1$

From the fact that

$$\lim_{d \rightarrow 0} \gamma(u_d(x), a) = 0$$

and the properties of our weights it follows that $\exists d > 0$ s.t. $\gamma(u_d(x), a) < \gamma_1$ and

$$(\gamma(u_d(x), a), \zeta(u_d(x), b), c)$$

is realizable, and from step 1 of Theorem 4.6 and its modification at Theorem 4.11 it follows that the point (γ_1, ζ, c_1) is realizable.

The existence of a realizable point (γ, ζ_1, c_1) for every fixed $\zeta_1 > 0$ and $c_1 > 0$ is proved in the same way. □

Now we can define:

Definition 5.5. We define $\psi_{w_1(x), w_2(\omega)}^{\gamma^m}(\zeta^m, c)$, $\psi_{w_1(x), w_2(\omega)}^{\zeta^m}(\gamma^m, c)$ and $\psi_{w_1(x), w_2(\omega)}^m(\gamma^m, \zeta^m)$ on $D = (0, \infty) \times (0, \infty)$ by:

$$(46) \quad \psi_{w_1(x), w_2(\omega)}^{\gamma^m}(\zeta^m, c) = \inf \{ \gamma^m \mid (\gamma^m, \zeta^m, c) \in PB_{\gamma^m, \zeta^m} \}$$

$$(47) \quad \psi_{w_1(x), w_2(\omega)}^{\zeta^m}(\gamma^m, c) = \inf\{\zeta^m \mid (\gamma^m, \zeta^m, c) \in PB_{\gamma^m, \zeta^m}\}$$

$$(48) \quad \psi_{w_1(x), w_2(\omega)}^m(\gamma^m, \zeta^m) = \inf\{c \mid (\gamma^m, \zeta^m, c) \in PB_{\gamma^m, \zeta^m}\}$$

Basically we have just change our point of view concerning the general uncertainty principles and the following facts are easy to see:

a) The general uncertainty principles can be written also in the forms:

$$(49) \quad \gamma^m \geq \psi_{w_1(x), w_2(\omega)}^{\gamma^m}(\zeta^m, c)$$

and

$$(50) \quad \zeta^m \geq \psi_{w_1(x), w_2(\omega)}^{\zeta^m}(\gamma^m, c)$$

where $\psi_{w_1(x), w_2(\omega)}^{\gamma^m}(\zeta^m, c)$ and $\psi_{w_1(x), w_2(\omega)}^{\zeta^m}(\gamma^m, c)$ have the same properties as $\psi_{w_1(x), w_2(\omega)}(\gamma, \zeta)$ (see Theorem 4.11).

b) $\psi_{w_1(x), w_2(\omega)}^m(\gamma^m, \zeta^m) = \psi_{w_1(x), w_2(\omega)}(\gamma, \zeta)$.

c) If $w_1(x) = w_2(x)$ then $\psi_{w_1(x), w_2(\omega)}^{\gamma^m}(\sigma, c) = \psi_{w_1(x), w_2(\omega)}^{\zeta^m}(\sigma, c)$.

Now we will focus on homogeneous weights. We will show that $\forall m \in N$ the possibility body of order m is convex, and find explicitly the related general uncertainty principles.

When we use homogenous weights of degree k and the parameter set (γ^m, ζ^m, c) we will use the following notation:

The possible area of level c will be denoted by: $M_c^{k,m}$

The possibility body will be denoted by: PB_{γ^m, ζ^m}^k

Lemma 5.6. *If $w_2(\omega)$ is homogeneous of order k (i.e. $w_2(g\omega) = g^k w_2(\omega)$) then*

$$(51) \quad (p, q) \in M_1^{k,m} \text{ iff } (p, c_0^{-\frac{km}{2}} q) \in M_{c_0}^{k,m}$$

and

$$(52) \quad \psi_{w_1(x), w_2(\omega)}^{\zeta^m}(\gamma^m, c_0) = c_0^{-\frac{km}{2}} \psi_{w_1(x), w_2(\omega)}^{\zeta^m}(\gamma^m, 1)$$

If $w_1(x)$ is homogeneous of order k (i.e. $w_1(gx) = g^k w_1(x)$) then

$$(53) \quad (p, q) \in M_1^{k,m} \text{ iff } (c_0^{-\frac{km}{2}} p, q) \in M_{c_0}^{k,m}$$

and

$$(54) \quad \psi_{w_1(x), w_2(\omega)}^{\gamma^m}(\zeta^m, c_0) = c_0^{-\frac{km}{2}} \psi_{w_1(x), w_2(\omega)}^{\gamma^m}(\zeta^m, 1)$$

Proof. Since

$$(55) \quad \begin{aligned} \zeta^m(f, c_0) &= \frac{(\int_{-\infty}^{\infty} w_{2c_0}(\omega) |\hat{f}(\omega)|^2 d\omega)^{\frac{m}{2}}}{\|f\|} = \frac{(\int_{-\infty}^{\infty} \frac{1}{c_0^k} w_2(\omega) |\hat{f}(\omega)|^2 d\omega)^{\frac{m}{2}}}{\|f\|} = \\ &= c_0^{-\frac{km}{2}} \zeta^m(f, 1) \end{aligned}$$

and since without loss of generality, we can take $a = 1, b = c_0, c_0 = ab$, for calculating the map of level c_0 and we can take $a = 1, b = 1, 1 = ab$ for calculating the map of level 1, we have:

$$M_{c_0}^{k,m} = \bigcup_{f \in L^2} (\gamma^m(f, 1), \zeta^m(f, c_0)) = \bigcup_{f \in L^2} (\gamma^m(f, 1), c_0^{-\frac{km}{2}} \zeta^m(f, 1))$$

we get that $(p, q) \in M_1^{k,m}$ iff $(p, c_0^{-\frac{km}{2}} q) \in M_{c_0}^{k,m}$ and (52) follows. The second part of the lemma is done in the same way. \square

Theorem 5.7. *If $w_1(x) = w_2(x)$ and the weights $w_1(x)$ and $w_2(x)$ are homogeneous of degree $k \in N$ then*

$$(56) \quad \psi_{w_1(x), w_2(\omega)}^m(\gamma^m, \zeta^m) = C(\gamma^m \zeta^m)^{\frac{-2}{km}}$$

The sets PB_{γ^m, ζ^m}^k $m \in N$ are convex. Either all of them are open or all of them are closed

Proof. First we show that $\forall c_0 \in R^+$ a point $(\gamma, 1, c_0) \in PB_{\gamma_1, \zeta_1}$ iff the point $(\frac{1}{\zeta_0} \gamma, \zeta_0, c_0) \in PB_{\gamma_1, \zeta_1}$:

$(\gamma, 1, c_0) \in PB_{\gamma_1, \zeta_1}$ implies that $\exists f \in L^2$ such that for $a = c_0, b = 1$ we have:

$$(57) \quad \gamma(f, c_0) = \frac{(\int_{-\infty}^{\infty} w_{1c_0}(x) |f(x)|^2 dx)^{\frac{1}{2}}}{\|f\|} = \frac{c_0^{\frac{-k}{2}} (\int_{-\infty}^{\infty} w_{11}(x) |f(x)|^2 dx)^{\frac{1}{2}}}{\|f\|} = \gamma$$

and

$$(58) \quad \zeta(f, 1) = \frac{(\int_{-\infty}^{\infty} w_{21}(\omega) |\hat{f}(\omega)|^2 d\omega)^{\frac{1}{2}}}{\|f\|} = 1$$

then for the same function $f \in L^2$ and $a = c_0 \zeta_0^{\frac{2}{k}}$, $b = \zeta_0^{\frac{2}{k}}$ we have:

$$(59) \quad \gamma(f, c_0 \zeta_0^{\frac{2}{k}}) = \frac{(\int_{-\infty}^{\infty} \frac{1}{(c_0 \zeta_0^{\frac{2}{k}})^k} w_{11}(x) |f(x)|^2 dx)^{\frac{1}{2}}}{\|f\|} = \frac{1}{\zeta_0} \gamma(f, c_0) = \frac{1}{\zeta_0} \gamma$$

and

$$(60) \quad \zeta(f, \zeta^{-\frac{2}{k}}) = \frac{(\int_{-\infty}^{\infty} w_{2\frac{-2}{k}}(\omega) |\hat{f}(\omega)|^2 d\omega)^{\frac{1}{2}}}{\|f\|} = \zeta_0 \zeta(f, 1) = \zeta_0$$

which implies that $(\frac{1}{\zeta_0} \gamma, \zeta_0, c_0) \in PB_{\gamma^1, \zeta^1}$.

The other direction is done in a similar way.

From the definition of $\psi_{w_1(x), w_2(\omega)}^{\gamma^1}(\zeta^1, c)$ (5.5) we get that

$$\psi_{w_1(x), w_2(\omega)}^{\gamma^1}(\zeta_0, c_0) = \frac{1}{\zeta_0} \psi_{w_1(x), w_2(\omega)}^{\gamma^1}(1, c_0)$$

and that if the infimum is attained in one point, say (ζ^*, c_0) , then it is attained for every (ζ, c_0) , $\zeta \in R^+$.

From lemma 5.6 we get

$$(61) \quad \psi_{w_1(x), w_2(\omega)}^{\gamma^1}(\zeta, c) = c^{\frac{-k}{2}} \psi_{w_1(x), w_2(\omega)}^{\gamma^1}(\zeta, 1) = \psi_{w_1(x), w_2(\omega)}^{\gamma^1}(1, 1) \frac{1}{\zeta} c^{\frac{-k}{2}}$$

and that if the infimum is attained for all the points (ζ, c_0) , $\zeta \in R^+$, then it is attained for all the points $(\zeta, 1)$, $\zeta \in R^+$, and therefore (using Lemma 5.6 again) it is attained for every point $(\zeta, c) \in R^{2+}$.

From the explicit formula for $\psi_{w_1(x), w_2(\omega)}^{\gamma^1}(\zeta, c)$ we see that the graph of $\psi_{w_1(x), w_2(\omega)}^{\gamma^1}(\zeta, c)$ is concave (as a multiplication of two concave functions; the proof of this simple fact is similar to the case of a sum of two concave functions that appears as part of the proof of Theorem 3.9) and that it is equal to $\partial PB_{\gamma^1, \zeta^1}$. From what we have showed above we get that PB_{γ^1, ζ^1} is closed iff $\exists v \in R^3$ s.t. $v \in \partial PB_{\gamma^1, \zeta^1} \cap PB_{\gamma^1, \zeta^1}$.

Thus we get that PB_{γ^1, ζ^1} is convex and either open or closed.

From convexity of PB_{γ^1, ζ^1} and the fact that $\psi_{w_1(x), w_2(\omega)}^{\gamma^1}(\zeta^1, c)$,

$\psi_{w_1(x), w_2(\omega)}^{\zeta^1}(\gamma^1, c)$ and $\psi_{w_1(x), w_2(\omega)}^1(\gamma^1, \zeta^1)$ are defined on $(0, \infty) \times (0, \infty)$ (see Lemma 5.4 and Definition 5.5) we get that their graphs coincide and equal to:

$$(62) \quad \partial PB_{\gamma^1, \zeta^1} = \{(\gamma, \zeta, c) \in R^{3+} | \gamma = \psi_{w_1(x), w_2(\omega)}^{\gamma^1}(1, 1) \frac{1}{\zeta} c^{\frac{-k}{2}}\}$$

and we get

$$\psi_{w_1(x), w_2(\omega)}^1(\gamma^1, \zeta^1) = C(\gamma\zeta)^{\frac{-2}{k}}$$

substituting $\gamma^1 = 1$ and $\zeta^1 = 1$ we find that $C = \psi_{w_1(x), w_2(\omega)}^1(1, 1)$

Thus we have

$$\psi_{w_1(x), w_2(\omega)}^1(\gamma^1, \zeta^1) = \psi_{w_1(x), w_2(\omega)}^1(1, 1)(\gamma\zeta)^{\frac{-2}{k}}$$

and

$$(63) \quad \psi_{w_1(x), w_2(\omega)}^m(\gamma^m, \zeta^m) = \psi_{w_1(x), w_2(\omega)}^1(1, 1)(\gamma^m \zeta^m)^{\frac{-2}{km}}$$

From the relation between the possibility bodies (45) and from (63) it is easy to see that: $\forall k \in N$ the sets, PB_{γ^m, ζ^m}^k $m \in N$ are convex; they are open iff the set PB_{γ^1, ζ^1}^k is open and closed iff the set PB_{γ^1, ζ^1}^k is closed. \square

When we take $k = 2$ and $m = 1$ in Theorem 5.7 we get the Heisenberg-Pauli-Weyl general uncertainty principle

$$\psi_{w_1(x), w_2(\omega)}^1(\gamma^1, \zeta^1) = C(\gamma\zeta)^{-1}$$

In Subsection 4.1 we also managed to compute the constant $C = \frac{1}{4\pi}$ for this special case (see (33) and (37)).

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